Classical Electrodynamics

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Lumped circuit concepts such as the resistance and reactance of a two-terminal linear network occur in many applications, even in circumstances where the size of the system is comparable to the free-space wavelength, for cxample, for a resonant antenna. It is uscful therefore to have a gencral definition based on field concepts. This follows from consideration of Poynting's theorem for harmonic time variation of the ficlds. We assume that all ficlds and sources have a time dependence $e^{-i\omega t}$, so that we write

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}[\mathbf{E}(\mathbf{x})e^{-i\omega t}] = \tfrac{1}{2}[\mathbf{E}(\mathbf{x})e^{-i\omega t} + \mathbf{E}^*(\mathbf{x})e^{i\omega t}] \qquad (6.128)$$

The field $\mathbf{E}(\mathbf{x})$ is in general complex, with a magnitude and phase that change with position. For product forms, such as $\mathbf{J}(\mathbf{x},t) \cdot \mathbf{E}(\mathbf{x},t)$, we have

$$\begin{aligned} \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t) &= \tfrac{1}{4}[\mathbf{J}(\mathbf{x})e^{-i\omega t} + \mathbf{J}^*(\mathbf{x})e^{i\omega t}] \cdot [\mathbf{E}(\mathbf{x})e^{-i\omega t} + \mathbf{E}^*(\mathbf{x})e^{i\omega t}] \\ &= \tfrac{1}{2}\,\mathrm{Re}[\mathbf{J}^*(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})e^{-2i\omega t}] \end{aligned} \qquad (6.129)$$

For time averages of products, the convention is therefore to take one-half of the real part of the product of one complex quantity with the complex conjugate of the other.

For harmonic ficlds the Maxwell equations become

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \qquad & \nabla \times \mathbf{E} - i\omega\mathbf{B} &= 0 \\ \nabla \cdot \mathbf{D} &= \rho, \qquad & \nabla \times \mathbf{H} + i\omega\mathbf{D} &= \mathbf{J} \end{aligned} \qquad (6.130)$$

where all the quantities arc complex functions of $\mathbf{x}$, according to the right-hand side of (6.128). Instead of (6.103) we consider the volume integral

$$\frac{1}{2}\int_V \mathbf{J}^* \cdot \mathbf{E}\, d^3x$$

whose real part gives the time-averaged rate of work done by the fields in the volume V. In a development strictly paralleling the steps from (6.103) to (6.107), we have

$$\begin{aligned} \frac{1}{2}\int_V \mathbf{J}^* \cdot \mathbf{E}\, d^3x &= \frac{1}{2}\int_V \mathbf{E} \cdot [\nabla \times \mathbf{H}^* - i\omega\mathbf{D}^*]\, d^3x \\ &= \frac{1}{2}\int_V [-\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) - i\omega(\mathbf{E} \cdot \mathbf{D}^* - \mathbf{B} \cdot \mathbf{H}^*)]\, d^3x \end{aligned} \qquad (6.131)$$