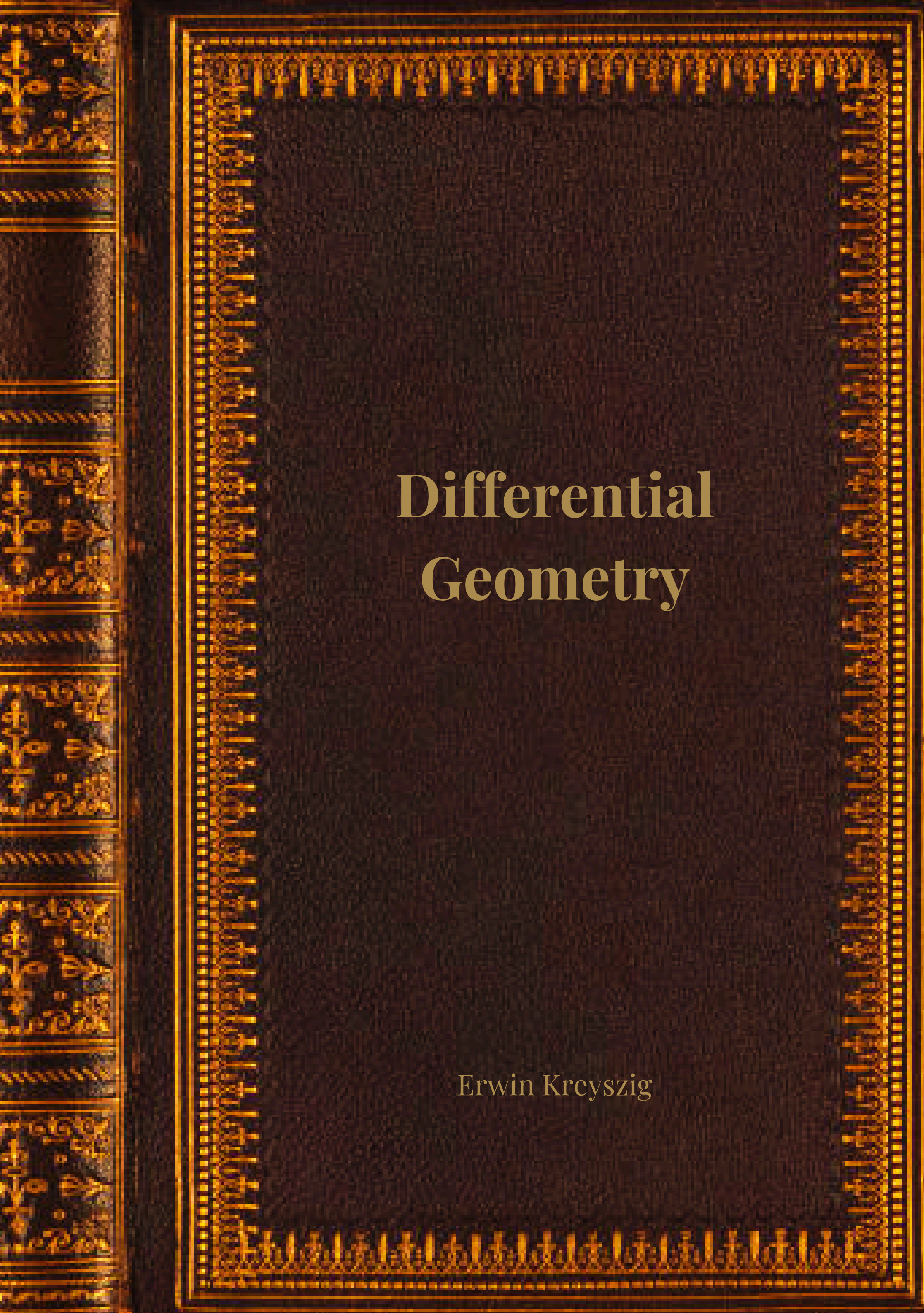

Differential Geometry

Erwin Kreyszig

We determine $\mathbf{x}_{\alpha\beta\lambda}$. From (45.14) we obtain

$$\mathbf{x}_{\alpha\beta\lambda} = \frac{\partial \Gamma_{\alpha\beta}{}^{\kappa}}{\partial u^{\lambda}} \mathbf{x}_{\kappa} + \Gamma_{\alpha\beta}{}^{\kappa} \mathbf{x}_{\kappa\lambda} + \frac{\partial b_{\alpha\beta}}{\partial u^{\lambda}} \mathbf{n} + b_{\alpha\beta} \mathbf{n}_{\lambda}$$

and from this, in consequence of (45.1) and (45.14),

$$\mathbf{x}_{\alpha\beta\lambda} = \frac{\partial \Gamma_{\alpha\beta}{}^{\kappa}}{\partial u^{\lambda}} \mathbf{x}_{\kappa} + \Gamma_{\alpha\beta}{}^{\kappa} (\Gamma_{\kappa\lambda}{}^{\sigma} \mathbf{x}_{\sigma} + b_{\kappa\lambda} \mathbf{n}) - b_{\alpha\beta} b_{\lambda}{}^{\tau} \mathbf{x}_{\tau} + \frac{\partial b_{\alpha\beta}}{\partial u^{\lambda}} \mathbf{n}.$$

If we exchange some terms and change some summation indeces we obtain

$$(46.1) \qquad \mathbf{x}_{\alpha\beta\lambda} = \left[\frac{\partial \Gamma_{\alpha\beta}{}^{\sigma}}{\partial u^{\lambda}} + \Gamma_{\alpha\beta}{}^{\kappa} \Gamma_{\kappa\lambda}{}^{\sigma} - b_{\alpha\beta} b_{\lambda}{}^{\sigma}\right] \mathbf{x}_{\sigma} + \left[\Gamma_{\alpha\beta}{}^{\rho} b_{\rho\lambda} + \frac{\partial b_{\alpha\beta}}{\partial u^{\lambda}}\right] \mathbf{n}.$$

If we exchange β and λ we find

$$(46.1') \qquad \mathbf{x}_{\alpha\lambda\beta} = \left[\frac{\partial \Gamma_{\alpha\lambda}{}^{\sigma}}{\partial u^{\beta}} + \Gamma_{\alpha\lambda}{}^{\kappa} \Gamma_{\kappa\beta}{}^{\sigma} - b_{\alpha\lambda} b_{\beta}{}^{\sigma}\right] \mathbf{x}_{\sigma} + \left[\Gamma_{\alpha\lambda}{}^{\rho} b_{\rho\beta} + \frac{\partial b_{\alpha\lambda}}{\partial u^{\beta}}\right] \mathbf{n}.$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{n}$ are linearly independent. In order that (46.1) be equal to (46.1'), that is,

$$(46.2) \qquad \mathbf{x}_{\alpha\beta\lambda} - \mathbf{x}_{\alpha\lambda\beta} = 0$$

in this difference, the coefficient of each of those three vectors must vanish. This means in the case of $\mathbf{n}$ that

$$(46.3) \qquad \Gamma_{\alpha\beta}{}^{\rho} b_{\rho\lambda} - \Gamma_{\alpha\lambda}{}^{\rho} b_{\rho\beta} + \frac{\partial b_{\alpha\beta}}{\partial u^{\lambda}} - \frac{\partial b_{\alpha\lambda}}{\partial u^{\beta}} = 0$$

must hold. These relations are called the *formulae of Mainardi-Codazzi*. They already occur implicitly the famous 'Disquisitiones' of Gauss [1]. G. Mainardy [1] published relations from which follow the formulae (46.3) which are due to D. Codazzi [1].

(46.3) means

$$\frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} - \Gamma_{12}{}^{1} b_{11} + (\Gamma_{11}{}^{1} - \Gamma_{12}{}^{2}) b_{12} + \Gamma_{11}{}^{2} b_{22} = 0,$$