The Feynman Lectures on Physics

Feynman - Leighton - Sands

Now let's find the distribution in momentum. Let's let $\phi(p)$ stand for the amplitude to find the electron with the momentum p,

$$\phi(p) \equiv \langle \text{mom}\, p \,|\, \psi \rangle. \tag{16.27}$$

Substituting Eq. (16.25) into Eq. (16.24) we get

$$\phi(p) = \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \cdot K e^{-x^2/4\sigma^2}\, dx. \tag{16.28}$$

The integral can also be rewritten as

$$K e^{-p^2\sigma^2/\hbar^2} \int_{-\infty}^{+\infty} e^{-(1/4\sigma^2)(x+2ip\sigma^2/\hbar)^2}\, dx. \tag{16.29}$$

We can now make the substitution $u = x + 2ip\sigma^2/\hbar$, and the integral is

$$\int_{-\infty}^{+\infty} e^{-u^2/4\sigma^2}\, du = 2\sigma\sqrt{\pi}. \tag{16.30}$$

(The mathematicians would probably object to the way we got there, but the result is, nevertheless, correct.)

$$\phi(p) = (8\pi\sigma^2)^{1/4} e^{-p^2\sigma^2/\hbar^2}. \tag{16.31}$$

We have the interesting result that the amplitude function in p has precisely the same mathematical form as the amplitude function in x; only the width of the Gaussian is different. We can write this as

$$\phi(p) = (\eta^2/2\pi\hbar^2)^{-1/4} e^{-p^2/4\eta^2}, \tag{16.32}$$

where the half-width η of the p-distribution function is related to the half-width σ of the x-distribution by

$$\eta = \frac{\hbar}{2\sigma}. \tag{16.33}$$

Our result says: if we make the width of the distribution in x very small by making σ small, η becomes large and the distribution in p is very much spread out. Or, conversely: if we have a narrow distribution in p, it must correspond to a spread-out distribution in x. We can, if we like, consider η and σ to be